Control benchmark of a Hydro Power Plant

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Version 1 - July 8, 2011

Abstract

In this report we describe a control problem which can be used as a benchmark for control algorithms for distributed systems.

1 System overview

The system we consider is a hydro power plant composed by several subsystems connected together.

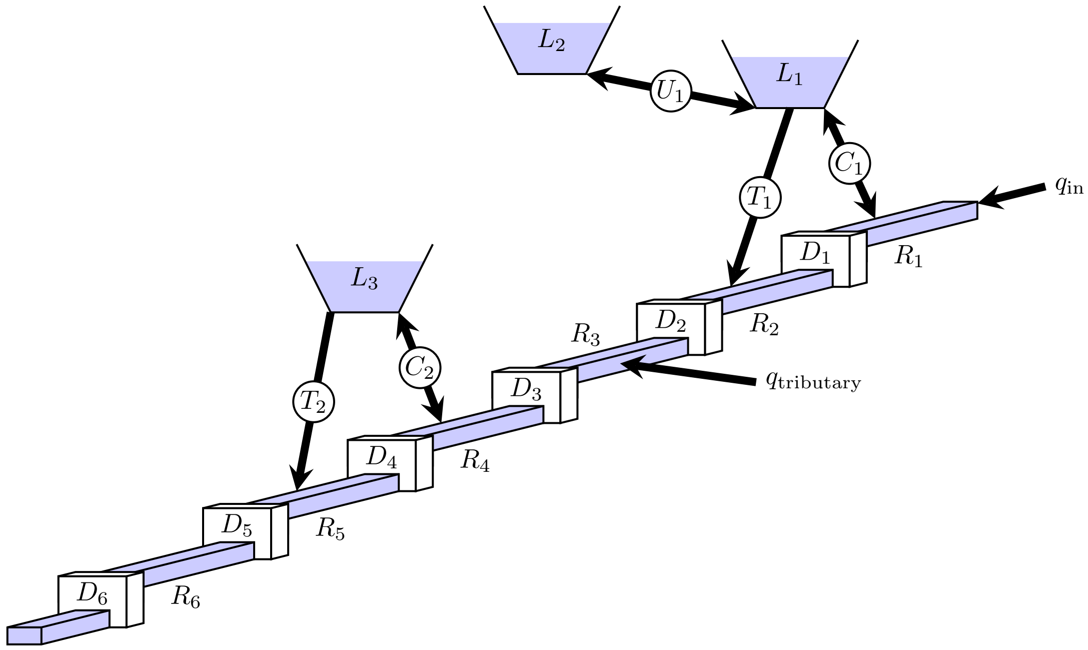


Figure 1: Overview of the hydro power plant.

Figure 1 gives an overview of the system which is composed by 3 lakes (L_1, L_2 and L_3) and a river which is divided in 6 reaches (R_1, R_2, R_3, R_4, R_5 and R_6) which terminate with dams equipped with turbines for power production (D_1, D_2, D_3, D_4, D_5 and D_6). The lakes and the river reaches are connected by a duct (U_1), ducts equipped with a turbine (T_1 and T_2) and ducts equipped with a pump and a turbine (C_1 and C_2). The river is fed by the flows q_{in} and $q_{\mathrm{tributary}}$.

In the following sections we shall provide a model for all the subsystems. The last sections contain the control specification and all the data needed to implement the model.

To simplify the system modeling we make the following assumptions:

- the ducts are connected at the bottom of the lakes (or to the bottom of the river bed);
- the cross section of the reaches and of the lakes is rectangular;
- the width of the reaches varies linearly along the reaches;
- the river bed slope is constant along every reach.

2 System modelling

2.1 Reach model

The model of the reaches is based on the one-dimensional Saint Venant partial differential equation:

$$\begin{cases} \dfrac{\partial q(t,z)}{\partial z} + \dfrac{\partial s(t,z)}{\partial t} = 0 \\ \dfrac{1}{g}\dfrac{\partial}{\partial t}\left(\dfrac{q(t,z)}{s(t,z)}\right) + \dfrac{1}{2g}\dfrac{\partial}{\partial z}\left(\dfrac{q^2(t,z)}{s^2(t,z)}\right) + \dfrac{\partial h(t,z)}{\partial z} + I_{\mathrm{f}}(t,z) - I_0(z) = 0 \end{cases} \tag{1}$$

The two equations in (1) express the mass and momentum balance. The variables represent the following quantities:

- z is the spatial variable which increases along the flow main direction;
- $q(t,z)$ is the river flow (or discharge) at time t and space coordinate z;
- $s(t,z)$ is the wetted surface;
- $h(t,z)$ is the water level w.r.t. the river bed;
- g is the gravitational acceleration;
- $I_f(t,z)$ is the friction slope;
- $I_0(z)$ is the river bed slope.

Assuming the cross section of the river is rectangular we can write the following equations:

$$s(t,z) = w(z)h(t,z) \tag{2}$$

and

$$I_f(t,z) = \frac{q(t,z)^2\left(w(z)+2h(t,z)\right)^{4/3}}{k_{\mathrm{str}}^2\left(w(z)h(t,z)\right)^{10/3}} \tag{3}$$

where $w(z)$ is the river width and k_{str} is the Gauckler-Manning-Strickler coefficient[1].

To take into account lateral inflows, the first equation in (1) which expresses the mass balance can be modified as follows

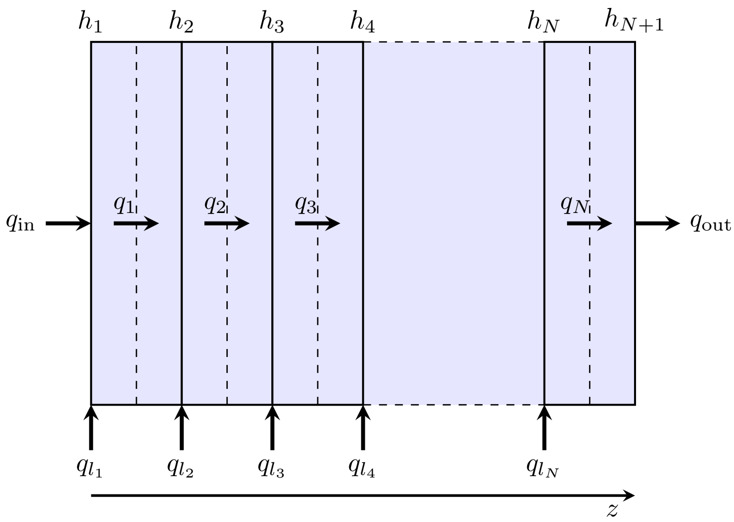
$$\frac{\partial q(t,z)}{\partial z} + \frac{\partial s(t,z)}{\partial t} = q_{\mathrm{l}}(z) \tag{4}$$

where $q_{\mathrm{l}}(z)$ is the lateral inflow per space unit.

2.1.1 Discretized model

The partial differential equation (1) can be converted into an ordinary differential equation with the method of lines. Divide the reach into N cells of length dz and denote by $q_i(t)$ the value of the discharge in the middle of the cell i and by $h_i(t)$ the value of the water level at the beginning of cell i. h_{N+1} represents the water level at the end of the reach.

[1] The Gauckler-Manning-Strickler coefficient changes accordingly to the kind of river bed surface. In the model we developed k_{str} is constant along the river.



Denoting by $q_{\text{in}}(t)$ and $q_{\text{out}}(t)$ the water inflow at the beginning of the reach and the water outflow at the end of the reach, we obtain the following set of ordinary differential equations (time dependencies are omitted)

$$\left\{\begin{array}{lcl} \dfrac{\partial h_1}{\partial t} & = & -\dfrac{1}{w_1}\dfrac{q_1 - q_{\text{in}} - q_{l_1}}{dz/2} \\ \dfrac{\partial q_1}{\partial t} & = & \dfrac{q_1}{w_1 h_1}\dfrac{q_{l_1}}{dz/2} - \dfrac{2q_1}{w_1 h_1}\dfrac{q_1 - q_{\text{in}}}{dz/2} + \left[\dfrac{1}{w_1}\left(\dfrac{q_1}{h_1}\right)^2 - g w_1 h_1\right]\dfrac{h_2 - h_1}{dz} + \\ & & + g w_1 h_1 I_0 - g w_1 h_1 \left[\dfrac{q_1^2\,(w_1 + 2h_1)^{4/3}}{k_{\text{str}}^2\,(w_1 h_1)^{10/3}}\right] \end{array}\right.$$

$$\left\{\begin{array}{lcl} \dfrac{\partial h_i}{\partial t} & = & -\dfrac{1}{w_i}\dfrac{q_i - q_{i-1} - q_{l_i}}{dz} \\ \dfrac{\partial q_i}{\partial t} & = & \dfrac{q_i}{w_i h_i}\dfrac{q_{l_i}}{dz} - \dfrac{2q_i}{w_i h_i}\dfrac{q_i - q_{i-1}}{dz} + \left[\dfrac{1}{w_i}\left(\dfrac{q_i}{h_i}\right)^2 - g w_i h_i\right]\dfrac{h_{i+1} - h_i}{dz} + \qquad i = 2, \dots, N \\ & & + g w_i h_i I_0 - g w_i h_i \left[\dfrac{q_i^2\,(w_i + 2h_i)^{4/3}}{k_{\text{str}}^2\,(w_i h_i)^{10/3}}\right] \end{array}\right. \tag{5}$$

$$\frac{\partial h_{N+1}}{\partial t} = -\frac{1}{w_{N+1}}\frac{q_{\text{out}} - q_N}{dz/2}$$

where w_i represents the river width at the beginning of cell i, w_{N+1} represents the river width at the end of the reach and q_{l_i} is the total lateral inflow of cell i. The river bed slope I_0 is assumed to be constant. Since the width of the reaches changes linearly, the values of w_1 and w_{N+1} are provided in the model data while

$$w_i = w_1 + \frac{(i-1)(w_{N+1} - w_1)}{N}. \tag{6}$$

Remark 1. *Notice that distance between the beginning of the reaches and the lateral inflow points are given in the last section. They are denoted as* $L_{\text{tributary}}$, L_{C_1}, L_{T_1}, L_{C_2} *and* L_{T_2}.

2.2 Lake model

Denote by $q_{\text{in}}(t)$ and $q_{\text{out}}(t)$ the water inflow and outflow of the lake under consideration, respectively. The volume of water inside the lake varies accordingly to the following equation

$$\frac{\partial v(t)}{\partial t} = q_{\text{in}}(t) - q_{\text{out}}(t). \tag{7}$$

Since the cross section of the lake is assumed to be rectangular, (7) can be equivalently expressed as

$$\frac{\partial h(t)}{\partial t} = \frac{q_{\text{in}}(t) - q_{\text{out}}(t)}{S}, \tag{8}$$

where $h(t)$ is the water level and S is the lake surface area.

2.3 Duct model

The flow inside the duct U_1 can be modeled using Bernoulli's law. Assuming the duct section is much smaller than the lake surface, the flow from lake L_1 to lake L_2 can be expressed as

$$q_{U_1}(t) = S_{U_1} \operatorname{sign}(h_{L_2}(t) - h_{L_1}(t) + h_{U_1})\sqrt{2g|h_{L_2}(t) - h_{L_1}(t) + h_{U_1}|}, \tag{9}$$

where h_{L_1} and h_{L_2} are the water levels for lakes L_1 and L_2, h_{U_1} is the height difference of the duct, S_{U_1} is the section of the duct and g is the gravitational acceleration.

Denoting $x = h_{L_2}(t) - h_{L_1}(t) + h_{U_1}$, equation (9) can be written as $S_{U_1}\sqrt{2g}\operatorname{sign}(x)\sqrt{|x|}$. The function $\operatorname{sign}(x)\sqrt{|x|}$ is not differentiable for $x = 0$. The following approximation can be used to make the function $q_{U_1}(t)$ differentiable

$$\operatorname{sign}(x)\sqrt{|x|} \approx \frac{x}{(x^2 + \epsilon^4)^{1/4}}.$$

Notice that for $\epsilon = 0$ the two functions are equivalent, while keeping ϵ small we obtain a good approximation ($\frac{1}{\epsilon}$ corresponds to the derivative of the approximation at $x = 0$).

2.4 Turbine model

For every turbine we assume that we can control directly the turbine discharge. The power produced is given by the following equation

$$p_{\mathrm{t}}(t) = k_{\mathrm{t}} q_{\mathrm{t}}(t) \Delta h_{\mathrm{t}}(t), \tag{10}$$

where k_{t} is the turbine coefficient, $q_{\mathrm{t}}(t)$ is the turbine discharge and $\Delta h_{\mathrm{t}}(t)$ is the turbine head.

2.5 Pump model

Pumps can be modeled similarly to turbines. The power absorbed by a pump is given by

$$p_{\mathrm{p}}(t) = k_{\mathrm{p}} q_{\mathrm{p}}(t) \Delta h_{\mathrm{p}}(t), \tag{11}$$

where k_{p} is the pump coefficient, $q_{\mathrm{p}}(t)$ is the pump discharge and $\Delta h_{\mathrm{p}}(t)$ is the pump head.

2.6 Modelling of ducts equipped with a turbine and a pump

The ducts C_1 and C_2 are equipped with a pump and a turbine and therefore we can use equations (10) and (11) to express the amount of power generated or absorbed. However, the turbines and the pumps cannot function together and this should be expressed in the optimal control problems (OCPs) formulated using the hydro power plant. Depending on the OCP formulation and the method used to solve the problem different models can be used. In the remainder of this section we illustrate some possibilities in modelling C_1 (the same model can be used for C_2). We assume that the flow can be determined by the controller.

2.6.1 Discontinuous model

Denote by $q_{C_1}(t)$ the flow through duct C_1. We assume that:

- $q_{C_1}(t) \geq 0$ when C_1 functions as a turbine;
- $q_{C_1}(t) < 0$ when C_1 functions as a pump.

Notice that by using this convention we do not need to express explicitly that C_1 can function as a turbine or a pump alternatively. The power produced can be expressed as

$$p_{C_1}(t) = k_{C_1}(q_{C_1}(t)) q_{C_1}(t) \Delta h_{C_1}(t), \tag{12}$$

where $\Delta h_{C_1}(t)$ it the duct head which depends on the water level in lake L_1 and reach R_1 and

$$k_{C_1}(q_{C_1}(t)) = \begin{cases} k_{\mathrm{t}_{C_1}} & \text{when } q_{C_1}(t) \geq 0 \\ k_{\mathrm{p}_{C_1}} & \text{when } q_{C_1}(t) < 0 \end{cases}, \tag{13}$$

($k_{\mathrm{t}_{C_1}}$ is the turbine coefficient and $k_{\mathrm{p}_{C_1}}$ is the pump coefficient). The flow in C_1 is limited:

$$q_{C_1}(t) \in \left[-q_{C_{1p,max}}, -q_{C_{1p,min}}\right] \cup \left[q_{C_{1t,min}}, q_{C_{1t,max}}\right], \tag{14}$$

where the values $q_{C_{1p,max}}$, $q_{C_{1p,min}}$, $q_{C_{1t,min}}$ and $q_{C_{1t,max}}$ are positive (the subscript t stands for turbine, while p stands for pump).

Equation (13) and the constraint (14) make the model of the C_1 discontinuous and therefore not suitable for many control techniques.

2.6.2 Smoothed model

Equation (13) can be written as

$$k_{C_1}(q_{C_1}(t)) = \frac{1}{2}\left((1 + \mathrm{sign}(q_{C_1}(t)))\, k_{\mathrm{t}_{C_1}} + (1 - \mathrm{sign}(q_{C_1}(t)))k_{\mathrm{p}_{C_1}}\right) \tag{15}$$

and then made smooth using the following approximation

$$\mathrm{sign}(x) \approx \frac{x}{(x^2+\epsilon^2)^{1/2}} \tag{16}$$

(ϵ^{-1} corresponds to the derivative of the approximation at $x = 0$). The constraint (14) can be simplified by imposing

$$q_{C_1}(t) \in \left[-q_{C_{1p,max}}, q_{C_{1t,max}}\right]. \tag{17}$$

The previous model of C_1 is still highly nonlinear and may not be suitable for linear MPC applications.

2.6.3 Double flow model

Another simplified model can be obtained by introducing two separate variables to express the flow in C_1

- $q_{C_{1p}}(t)$: flow when C_1 is functioning as a pump;
- $q_{C_{1t}}(t)$: flow when C_1 is functioning as a turbine.

Assuming these new variables are both positive we can write

$$q_{C_1}(t) = q_{C_{1t}}(t) - q_{C_{1p}}(t) \tag{18}$$

and

$$p_{C_1}(t) = (k_{\mathrm{t}_{C_1}} q_{C_{1t}}(t) - k_{\mathrm{p}_{C_1}} q_{C_{1p}}(t))\Delta h_{C_1}(t). \tag{19}$$

The constraint (14) can be rewritten in terms of $q_{C_{1p}}(t)$ and $q_{C_{1t}}(t)$

$$q_{C_{1p}}(t) \in \left[q_{C_{1p,min}}, q_{C_{1p,max}}\right] \tag{20}$$

$$q_{C_{1t}}(t) \in \left[q_{C_{1t,min}}, q_{C_{1t,max}}\right]. \tag{21}$$

2.6.4 Relaxed model

When the power production is maximized (as in the profit maximization scenario porposed below), the following relaxation can be used

$$\begin{array}{l} p_{C_1}(t) \leq k_{\mathrm{t}_{C_1}} q_{C_{1t}}(t)\Delta h_{C_1}(t) \\ p_{C_1}(t) \leq k_{\mathrm{p}_{C_1}} q_{C_{1p}}(t)\Delta h_{C_1}(t) \end{array} \tag{22}$$

and

$$q_{C_1}(t) \in \left[-q_{C_{1p,max}}, q_{C_{1t,max}}\right] \tag{23}$$

This relaxation is meaningfull for power maximization since the value of $k_{\mathrm{p}_{C_1}} < k_{\mathrm{t}_{C_1}}$.

Remark 2. *Using any of the models in Sections 2.6.2, 2.6.3 or 2.6.4 introduces some approximations. In particular, the control inputs corresponding to the solution of an OCP using these simplified models may not respect constraint* (14)*. The control values achieved should be therefore modified.*

3 Subsystem partition

The system is partitioned into 8 subsystems.

3.1 Subsystem 1 ($L_1 + L_2 + U_1 + T_1 + C_1$)

Subsystem 1 is composed by lakes L_1 and L_2 and ducts U_1, T_1 and C_1. Duct C_1 can function as a pump or a turbine.

Define the following quantities:

- $h_{L_1}(t)$ is the water level in lake L_1;
- $h_{L_2}(t)$ is the water level in lake L_2;
- $q_{L_1}(t)$ is the water inflow for L_1 which takes into account rain, small tributaries, etc.;
- $q_{L_2}(t)$ is the water inflow for L_2 which takes into account rain, small tributaries, etc.;
- $q_{T_1}(t)$ is the water discharge going to turbine T_1 (control variable);
- $q_{C_1}(t)$ is the water discharge going through the duct C_1 (control variable);
- h_{T_1} is the height difference of duct T_1;
- h_{C_1} is the height difference of duct C_1;
- $h_{R_2,T_1}(t)$ is the water level in R_2 at the outflow point of duct T_1;
- $h_{R_1,C_1}(t)$ is the water level in R_1 at the outflow point of duct C_1;
- $k_{\mathrm{t}_{T_1}}$ is the turbine coefficient of T_1;
- $k_{\mathrm{t}_{C_1}}$ is the turbine coefficient of C_1;
- $k_{\mathrm{p}_{C_1}}$ is the pump coefficient of C_1
- $p_{S_1}(t)$ is the power produced by subsystem 1.

The equations governing the subsystem behavior can be derived using the equations illustrated in the previous section and setting

- lake L_1
$$\begin{aligned} q_{\mathrm{in}}(t) &= q_{L_1}(t) + q_{U_1}(t) \\ q_{\mathrm{out}}(t) &= q_{T_1}(t) + q_{C_1}(t) \end{aligned}$$
- lake L_2
$$\begin{aligned} q_{\mathrm{in}}(t) &= q_{L_2}(t) \\ q_{\mathrm{out}}(t) &= q_{U_1}(t) \end{aligned}$$
- turbine T_1
$$\Delta h_{\mathrm{t}}(t) = h_{T_1} + h_{L_1}(t) - h_{R_2,T_1}(t)$$
- combined turbine/pump C_1
$$\Delta h_{C_1}(t) = h_{C_1} + h_{L_1}(t) - h_{R_1,C_1}(t).$$

The variables of subsystem 1 are subject to the following constraints

$$\begin{aligned} &h_{L_{1\min}} \le h_{L_1}(t) \le h_{L_{1\max}} \\ &h_{L_{2\min}} \le h_{L_2}(t) \le h_{L_{2\max}} \\ &q_{T_{1\min}} \le q_{T_1}(t) \le q_{T_{1\max}} \\ &q_{C_1}(t) \in \left[-q_{C_{1\mathrm{p},\max}}, -q_{C_{1\mathrm{p},\min}}\right] \cup \left[q_{C_{1\mathrm{t},\min}}, q_{C_{1\mathrm{t},\max}}\right] \end{aligned}$$

3.2 Subsystem 2 (L_3 + T_2 + C_2)

Subsystem 2 is composed by lake L_3 and ducts T_2 and C_2.

Define the following quantities:

- $h_{L_3}(t)$ is the water level in lake L_3;
- $q_{L_3}(t)$ is the water inflow for L_3 which takes into account rain, small tributaries, etc.;
- $q_{T_2}(t)$ is the water discharge going to turbine T_2 (control variable);
- $q_{C_2}(t)$ is the water discharge going through the duct C_2. $q_{C_2}(t)$ is positive when C_2 functions as a pump (control variable);
- h_{T_2} is the height difference of duct T_2;
- h_{C_2} is the height difference of duct C_2;
- $h_{R_5,T_2}(t)$ is the water level in R_5 at the outflow point of duct T_2;
- $h_{R_4,C_2}(t)$ is the water level in R_4 at the outflow point of duct C_2;
- $k_{\mathrm{t}_{T_2}}$ is the turbine coefficient of T_2;
- $k_{\mathrm{t}_{C_2}}$ is the turbine coefficient of C_2;
- $k_{\mathrm{p}_{C_2}}$ is the pump coefficient of C_2;
- $p_{S_2}(t)$ is the power produced by subsystem 2.

The equations governing the subsystem behavior can be derived using equations (8)–(11) and setting

- lake L_3

$$\begin{aligned} q_{\mathrm{in}}(t) &= q_{L_3}(t) \\ q_{\mathrm{out}}(t) &= q_{T_2}(t) + q_{C_2}(t) \end{aligned}$$

- turbine T_2

$$\Delta h_{\mathrm{t}}(t) = h_{T_2} + h_{L_3}(t) - h_{R_5,T_2}(t)$$

- combined turbine/pump C_2

$$\Delta h_{C_2}(t) = h_{C_2} + h_{L_3}(t) - h_{R_4,C_2}(t).$$

The variables of subsystem 2 are subject to the following constraints

$$\begin{aligned} &h_{L_{3\min}} \le h_{L_3}(t) \le h_{L_{3\max}} \\ &q_{T_{2\min}} \le q_{T_2}(t) \le q_{T_{2\max}} \\ &q_{C_2}(t) \in \left[-q_{C_{2\mathrm{p},\max}}, -q_{C_{2\mathrm{p},\min}}\right] \cup \left[q_{C_{2\mathrm{t},\min}}, q_{C_{2\mathrm{t},\max}}\right] \end{aligned}$$

3.3 Subsystem 3 (R_1 + D_1), 4 (R_2 + D_2), 5 (R_3 + D_3), 6 (R_4 + D_4), 7 (R_5 + D_5), 8 (R_6 + D_6)

Subsystems 3, 4, 5, 6, 7, and 8 are composed by a reach and dam. Figure 2 represents the structure of the dams. All the flow going through the dams is used by the turbine to produce electricity. The head of the turbines inside the dams can be expressed as difference of the water level before and after the dam. Since the water level after dam D_6 is not part of the model we consider it constant ($h_{D_{6\mathrm{out}}}$). The constraints on the subsystem variables are

- subsystem 3

$$\begin{aligned} h_{R_{1\min}} &\le h_{R_1}(t) \le h_{R_{1\max}} \\ q_{D_{1\min}} &\le q_{D_1}(t) \le q_{D_{1\max}} \end{aligned}$$

  where $h_{R_1}(t)$ is the water level at the end of reach R_1 and $q_{D_1}(t)$ is the dam discharge which goes to the turbine (the control variable);

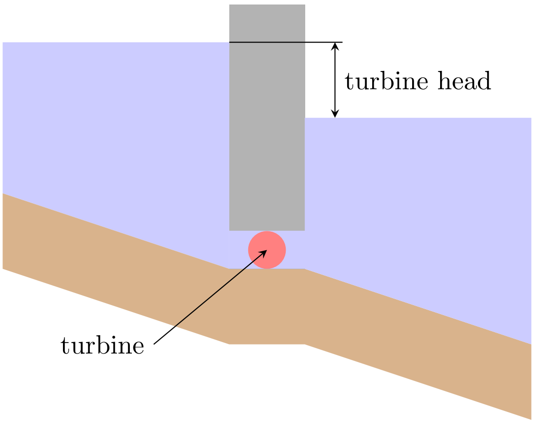


Figure 2: Dam configuration.

- subsystem 4

$$h_{R_{2\min}} \leq h_{R_2}(t) \leq h_{R_{2\max}}$$
$$q_{D_{2\min}} \leq q_{D_2}(t) \leq q_{D_{2\max}}$$

  where $h_{R_2}(t)$ is the water level at the end of reach R_2 and $q_{D_2}(t)$ is the dam discharge which goes to the turbine (the control variable);

- subsystem 5

$$h_{R_{3\min}} \leq h_{R_3}(t) \leq h_{R_{3\max}}$$
$$q_{D_{3\min}} \leq q_{D_3}(t) \leq q_{D_{3\max}}$$

  where $h_{R_3}(t)$ is the water level at the end of reach R_1 and $q_{D_3}(t)$ is the dam discharge which goes to the turbine (the control variable);

- subsystem 6

$$h_{R_{4\min}} \leq h_{R_4}(t) \leq h_{R_{4\max}}$$
$$q_{D_{4\min}} \leq q_{D_4}(t) \leq q_{D_{4\max}}$$

  where $h_{R_4}(t)$ is the water level at the end of reach R_4 and $q_{D_4}(t)$ is the dam discharge which goes to the turbine (the control variable);

- subsystem 7

$$h_{R_{5\min}} \leq h_{R_5}(t) \leq h_{R_{5\max}}$$
$$q_{D_{5\min}} \leq q_{D_5}(t) \leq q_{D_{5\max}}$$

  where $h_{R_5}(t)$ is the water level at the end of reach R_5 and $q_{D_5}(t)$ is the dam discharge which goes to the turbine (the control variable);

- subsystem 8

$$h_{R_{6\min}} \leq h_{R_6}(t) \leq h_{R_{6\max}}$$
$$q_{D_{6\min}} \leq q_{D_6}(t) \leq q_{D_{6\max}}$$

  where $h_{R_6}(t)$ is the water level at the end of reach R_6 and $q_{D_6}(t)$ is the dam discharge which goes to the turbine (the control variable).

4 Control test scenarios

We consider two test scenarios:

- in the first scenario the power output of the system should follow a given reference;
- in the second scenario the system profit should be maximized based on the available information on the hourly electricity price variations.

In the control test scenario we make the assumption that all the water inflows are constant ($q_{\text{in}}(t) = q_{\text{in}}$, $q_{\text{tributary}}(t) = q_{\text{tributary}}$, $q_{\text{in}_{L_1}}(t) = q_{\text{in}_{L_1}}$, $q_{\text{in}_{L_2}}(t) = q_{\text{in}_{L_2}}$, $q_{\text{in}_{L_3}}(t) = q_{\text{in}_{L_3}}$).

To simplify the description of the two optimal control problem formulations we define

- $x_i(t)$: state vector of subsystem i;
- $u_i(t)$: input vector of subsystem i;
- C_i: set describing the constraints for subsystem i;
- $p_i(x_i(t), u_i(t))$: power produced by subsystem i;

4.1 Power reference tracking

We assume that the power reference to be followed by the entire system is known 24 hours in advance. Therefore, the prediction horizon is set to 86400 seconds. The inputs of the system can be changed every 30 minutes.The input vectors $u_i(t)$ are constant in this time intervals.

The optimal control problem to be solved reads

$$\begin{aligned} \min_{x_i,u_i} \quad & \sum_{k=0}^{47} \gamma_k \int_{t_k}^{t_{k+1}} \left| p_r(t) - \sum_{i=1}^{8} p_i(x_i(t), u_i(t)) \right| \\ \text{s.t.} \quad & \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_8(t) \end{bmatrix} = f\left(\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_8(t) \end{bmatrix}, \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_8(t) \end{bmatrix} \right) \\ & (x_i(t), u_i(t)) \in C_i \quad i = 1, \ldots, 8 \\ & x_i(t) = x_{i,0} \quad i = 1, \ldots, 8 \end{aligned} \tag{24}$$

where $t_k = 1800k$, f is a function which represents the dynamics of the whole system. The function $p_r(t)$ is the given power reference (piecewise constant).

Remark 3. *Notice that when implementing this scenario the power should be expressed in MW (megawatts).*

4.2 Profit maximization

When maximizing the profit of the plant the electricity price is known 24 hours in advance and varies every hour. As in the power reference tracking scenario the inputs can be modified every 30 minutes.

The optimal control problem to be solved reads

$$\begin{aligned} \max_{x_i,u_i} \quad & \sum_{k=0}^{47} c_k \int_{t_k}^{t_{k+1}} \sum_{i=1}^{8} p_i(x_i(t), u_i(t)) dt + \sum_{i=1}^{8} c_{f,i}^T x_i(T) \\ \text{s.t.} \quad & \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_8(t) \end{bmatrix} = f\left(\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_8(t) \end{bmatrix}, \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_8(t) \end{bmatrix} \right) \\ & (x_i(t), u_i(t)) \in C_i \quad i = 1, \ldots, 8 \\ & x_i(t) = x_{i,0} \quad i = 1, \ldots, 8 \end{aligned} \tag{25}$$

where $t_k = 1800k$, c_k is the electricity price (in EURO/Wh) during the k-*th* half-hour, and $c_{f,i}$ is a vector pricing the water remaining in the system at the end of the control horizon.

4.3 How to compare different control techniques

Different control techniques require different assumption on the problem. If a new method should be tested, modifications can be made to the problem formulation and the model used. However, for a fair comparison, the performance should be evaluated using the model and the cost provided above.

In the data provided (see the next section and the MatLab files) a steady state for the system and the values of c_k and γ_k can be found. In the test the simulation should start from steady state. Since the values of c_k and γ_k are given for a period of 24 hours only, they can be considered periodic with a period of 24 hours.

5 Problem data

In the following table we can find the data necessary to implement the model. The data are grouped per subsystem. General data can be found at the end of the table.

Symbol	Description	Unit of measure	Value
	Subsystem 1		
S_{L_1}	Surface lake L_1	m^2	10×10^3
S_{L_2}	Surface lake L_2	m^2	5×10^3
$h_{L_{1\min}}$	Minimum lake water level L_1	m	10.5
$h_{L_{1\max}}$	Maximum lake water level L_1	m	13.5
$h_{L_{2\min}}$	Minimum lake water level L_2	m	5.5
$h_{L_{2\max}}$	Maximum lake water level L_2	m	8.5
$q_{\text{in}_{L_1}}$	Water inflow lake L_1	$m^3\ s^{-1}$	5
$q_{\text{in}_{L_2}}$	Water inflow lake L_2	$m^3\ s^{-1}$	5
h_{U_1}	Height difference duct U_1	m	5
S_{U_1}	Cross section duct U_1	m^2	6
$k_{t_{T_1}}$	Turbine coefficient T_1	$J\ m^{-4}$	8000
$k_{t_{C_1}}$	Turbine coefficient C_1	$J\ m^{-4}$	8000
$k_{p_{C_1}}$	Pump coefficient C_1	$J\ m^{-4}$	14000
h_{T_1}	Height difference duct T_1	m	223
h_{C_1}	Height difference duct C_1	m	200
$q_{T_{1\min}}$	Minimum turbine flow T_1	$m^3\ s^{-1}$	0
$q_{T_{1\max}}$	Maximum turbine flow T_1	$m^3\ s^{-1}$	20
$q_{C_{1\text{t},\min}}$	Minimum flow in C_1 in turbine mode	$m^3\ s^{-1}$	0
$q_{C_{1\text{t},\max}}$	Maximum flow in C_1 in turbine mode	$m^3\ s^{-1}$	10
$q_{C_{1\text{p},\min}}$	Minimum flow in C_1 in pump mode	$m^3\ s^{-1}$	0
$q_{C_{1\text{p},\max}}$	Maximum flow in C_1 in pump mode0.0	$m^3\ s^{-1}$	5
	Subsystem 2		
S_{L_3}	Surface lake L_3	m^2	10×10^3
$h_{L_{3\min}}$	Minimum lake water level L_3	m	6
$h_{L_{3\max}}$	Maximum lake water level L_3	m	9
$q_{\text{in}_{L_3}}$	Water inflow lake L_3	$m^3\ s^{-1}$	10
$k_{t_{T_2}}$	Turbine coefficient turbine T_2	$J\ m^{-4}$	8000
$k_{t_{C_2}}$	Turbine coefficient C_2	$J\ m^{-4}$	8000
$k_{p_{C_2}}$	Pump coefficient C_2	$J\ m^{-4}$	14000
h_{T_2}	Height difference duct C_2	m	233
h_{C_2}	Height difference duct T_2	m	250
$q_{T_{2\min}}$	Minimum turbine flow T_2	$m^3\ s^{-1}$	0
$q_{T_{2\max}}$	Maximum turbine flow T_2	$m^3\ s^{-1}$	20
$q_{C_{2\text{t},\min}}$	Minimum flow in C_2 in turbine mode	$m^3\ s^{-1}$	0
$q_{C_{2\text{t},\max}}$	Maximum flow in C_2 in turbine mode	$m^3\ s^{-1}$	10
$q_{C_{2\text{p},\min}}$	Minimum flow in C_2 in pump mode	$m^3\ s^{-1}$	0
$q_{C_{2\text{p},\max}}$	Maximum flow in C_2 in pump mode	$m^3\ s^{-1}$	5
	Subsystem 3		
$I_{0_{R_1}}$	Bed slope reach R_1	-	0.0025
L_{R_1}	Length reach R_1	m	10000
$w_{1_{R_1}}$	Width beginning reach R_1	m	30
$w_{N_{R_1}}$	Width end reach R_1	m	50
N_{R_1}	Number of cells used in the discretizartion of reach R_1	-	20
$k_{\text{t}_{D_1}}$	Turbine constant dam D_1	$J\ m^{-4}$	8000

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Symbol	Description	Unit of measure	Value
$h_{R_{1\min}}$	Minimum water level at dam D_1	m	14.5
$h_{R_{1\max}}$	Maximum water level at dam D_1	m	17.5
$q_{D_{1\min}}$	Minimum turbine flow dam D_1	m^3s^{-1}	5
$q_{D_{1\max}}$	Maximum turbine flow dam D_1	$m^3\ s^{-1}$	300
q_{in}	Water inflow reach R_1	m^3s^{-1}	200
L_{C_1}	Distance from the beginning of the reach to duct C_1	m	5000
	Subsystem 4		
$I0_{R_2}$	Bed slope of reach R_2	-	0.0015
L_{R_2}	Length reach R_2	m	8000
$w1_{R_2}$	Width beginning reach R_2	m	40
wN_{R_2}	Width end reach R_2	m	45
N_{R_2}	Number of cells used in the discretization of reach R_2	-	20
$k_{\text{t}_{D_2}}$	Turbine constant dam D_2	$J\ m^{-4}$	8000
$h_{R_{2\min}}$	Minimum water level at dam D_2	m	16.5
$h_{R_{2\max}}$	Maximum water level at dam D_2	m	19.5
$q_{D_{2\min}}$	Minimum turbine flow dam D_2	m^3s^{-1}	5
$q_{D_{2\max}}$	Maximum turbine flow dam D_2	$m^3\ s^{-1}$	300
L_{T_1}	Distance from the beginning of the reach to duct T_1	m	4000
	Subsystem 5		
$I0_{R_3}$	Bed slope reach R_3	-	0.002
L_{R_3}	Length reach R_3	m	6000
$w1_{R_3}$	Width beginning reach R_3	m	40
wN_{R_3}	Width end reach R_3	m	55
N_{R_3}	Number of cells used in the discretizartion of reach R_3	-	20
$k_{\text{t}_{D_3}}$	Turbine constant dam D_3	$J\ m^{-4}$	8000
$h_{R_{3\min}}$	Minimum water level at dam D_3	m	21.5
$h_{R_{3\max}}$	Maximum water level at dam D_3	m	24.5
$q_{D_{3\min}}$	Minimum turbine flow dam D_3	m^3s^{-1}	5
$q_{D_{3\max}}$	Maximum turbine flow dam D_3	$m^3\ s^{-1}$	300
$L_{\text{tributary}}$	Distance from the beginning of the reach to tributary inflow point	m	3000
$q_{\text{tributary}}$	Tributary inflow	$m^3\ s^{-1}$	30
	Subsystem 6		
$I0_{R_4}$	Bed slope of reach R_2	-	0.0015
L_{R_4}	Length reach R_2	m	8000
$w1_{R_4}$	Width beginning reach R_2	m	55
wN_{R_4}	Width end reach R_2	m	45
N_{R_4}	Number of cells used in the discretizartion of reach R_4	-	20
$k_{\text{t}_{D_4}}$	Turbine constant dam D_4	$J\ m^{-4}$	8000
$h_{R_{4\min}}$	Minimum water level at dam D_4	m	17.5
$h_{R_{4\max}}$	Maximum water level at dam D_4	m	20.5
$q_{D_{4\min}}$	Minimum turbine flow dam D_4	m^3s^{-1}	5
$q_{D_{4\max}}$	Maximum turbine flow dam D_4	$m^3\ s^{-1}$	300
L_{C_2}	Distance from the beginning of the reach to duct C_2	m	4000
	Subsystem 7		

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Symbol	Description	Unit of measure	Value
$I0_{R_5}$	Bed slope reach R_5	-	0.0025
L_{R_5}	Length reach R_5	m	6000
$w1_{R_5}$	Width beginning reach R_5	m	50
wN_{R_5}	Width end reach R_5	m	60
N_{R_5}	Number of cells used in the discretizartion of reach R_5	-	20
$k_{\mathrm{t}D_5}$	Turbine constant dam D_5	J m^{-4}	8000
$h_{R_{5\,\mathrm{min}}}$	Minimum water level at dam D_5	m	13.5
$h_{R_{5\,\mathrm{max}}}$	Maximum water level at dam D_5	m	16.5
$q_{D_{5\,\mathrm{min}}}$	Minimum turbine flow dam D_5	m^{3}s^{-1}	5
$q_{D_{5\,\mathrm{max}}}$	Maximum turbine flow dam D_5	m^3 s^{-1}	300
L_{T_2}	Distance from the beginning of the reach to duct T_2	m	3000
	Subsystem 8		
$I0_{R_6}$	Bed slope of reach R_6	-	0.002
L_{R_6}	Length reach R_6	m	8000
$w1_{R_6}$	Width beginning reach R_6	m	60
wN_{R_6}	Width end reach R_6	m	80
N_{R_6}	Number of cells used in the discretizartion of reach R_6	-	20
$k_{\mathrm{t}D_6}$	Turbine constant dam D_6	J m^{-4}	8000
$h_{R_{6\,\mathrm{min}}}$	Minimum water level at dam D_6	m	11.5
$h_{R_{6\,\mathrm{max}}}$	Maximum water level at dam D_6	m	14.5
$q_{D_{6\,\mathrm{min}}}$	Minimum turbine flow dam D_6	m^{3}s^{-1}	10
$q_{D_{6\,\mathrm{max}}}$	Maximum turbine flow dam D_6	m^3 s^{-1}	300
$h_{D_{6\,\mathrm{out}}}$	Water level after dam D_6	m	2
	General data		
k_{str}	Gauckler-Manning-Strickler coefficient	m$^{1/3}$ s^{-1}	30
g	Gravitational acceleration constant	m s^{-2}	9.81

The other data necessary for the implementation (initial condition of the system, power reference, electricity prices, ...) can be found in the files inside the folder `matlab_model` (check the file README.txt for instructions and some more details).